Quiz 10; Wednesday, November 1
MATH 110 with Professor Stankova
Section 112; 4-5 pm
GSI: Saad Qadeer

Solutions

You have 10 minutes to complete the quiz. Calculators are not permitted. Please include all relevant calculations and explanations (unless stated otherwise).

1. (12 points) Let T be a linear operator on a finite-dimensional vector space V. Prove that if $\text{rank}(T^m) = \text{rank}(T^{m+1})$ for some positive integer m, then $\text{Im}(T^m) = \text{Im}(T^{m+k})$ for any integer $k \geq 0$.

   We prove this by induction on k. For $k = 0$, the statement is trivial. Suppose it holds for some k, that is, $\text{Im}(T^m) = \text{Im}(T^{m+k})$. Any $v \in \text{Im}(T^m)$ can then be written as $v = T^{m+k}(w)$ for some $w \in V \Rightarrow v = T^k(T^m(w))$. As $\text{Im}(T^{m+1}) \subseteq \text{Im}(T^m)$ and $\text{rank}(T^m) = \text{rank}(T^{m+1})$, we have $\text{Im}(T^{m+1}) = \text{Im}(T^m)$ so we can write $T^m(w) = T^{m+1}(x)$ for some $x \in V$ so that

   $$v = T^k(T^{m+1}(x)) = T^{m+k+1}(x) \in \text{Im}(T^{m+k+1}).$$

   This establishes that $\text{Im}(T^m) \subseteq \text{Im}(T^{m+k+1})$. As $\text{Im}(T^{m+k+1}) \subseteq \text{Im}(T^m)$ as well, we conclude that $\text{Im}(T^m) = \text{Im}(T^{m+k+1})$ and so the result holds.

2. (1 + 1 + 1 points) Mark each statement as True or False. You do not need to show your work but a blank answer is worth 0 points and an incorrect answer is worth -1 point.

   (a) It is possible for a (nonzero) generalized eigenvector of a linear operator T to correspond to a scalar that is not an eigenvalue of T.

   False: If $(T - \lambda I)^{m-1}(v) \neq 0$ and $(T - \lambda I)^m(v) = 0$, then $(T - \lambda I)^{m-1}(v)$ is an eigenvector corresponding to eigenvalue λ so any scalar that leads to generalized eigenvectors must be an eigenvalue.

   (b) If T is a linear operator on V with λ as an eigenvalue, then the generalized eigenspace K_λ is T-invariant.

   True: Theorem 7.1(a).

   (c) Let T be a linear operator on an n-dimensional space V with only λ as an eigenvalue. If β is a basis for the generalized eigenspace K_λ, then $[T]_\beta$ is the Jordan block $J_n(\lambda)$.

   False: $[T]_\beta$ is a Jordan block if and only if β is a cycle of generalized eigenvectors.